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Branching rules for irreducible representations of E_8 into D_8†

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Abstract. The branching rules for irreducible representations of E_8 into D_8 are calculated using the knowledge of the Kronecker products for those two algebras. Tables of Kronecker products for both E_8 and D_8 algebras are also included.

1. Introduction

In the past few years, several models for grand unified theories based on exceptional Lie groups have been proposed (Gürsey and Ramond 1976, Gürsey and Sikivie 1976, Bars and Günaydin 1980). It has thus become necessary to study different properties of these exceptional algebras. Wybourne and Bowick (1977) and later Wybourne (1979) have developed a technique for calculating Kronecker products and branching rules for all exceptional algebras. However, the branching rules for the D_8 subalgebra of E_8 were not included. These are important, since the E_8 weights can be written in the same orthogonal basis as the D_8 weights, and this basis has proved to be useful, in particular when one wants to study the non-regular subalgebras of exceptional Lie algebras (Feldman *et al* 1982). Some of the branching rules have been given by King and Al-Qubanchi (1981) using the knowledge of weight multiplicities. Making use of the Kronecker products in E_8 and D_8, we have found King's results and extended the table of $E_8 \to D_8$ branching rules to include all E_8 irreducible representations (irreps) of dimension less than 76 271 625; this includes all irreps of length ≤ 18 and one of length 20. The length of a representation is defined below. Our method is discussed in § 2 together with some examples. Tables 1–4 list the E_8 and D_8 irreducible representations and the Kronecker products which are required for the main results which are presented in table 5.

2. E_8 to D_8 branching rules

A weight vector $\boldsymbol{\omega}'$ of an irreducible representation (irrep) is equivalent to another weight $\boldsymbol{\omega}$ if it can be expressed as $\boldsymbol{\omega}' = S_\alpha \boldsymbol{\omega}$, where S_α are elements of the Weyl group and $\boldsymbol{\alpha}$ are the roots of the algebra (King and Al-Qubanchi 1981). In an orthogonal basis, the roots of D_8 are

$$\pm\boldsymbol{\lambda}_i \pm \boldsymbol{\lambda}_j \qquad i, j = 1, 2, \ldots, 8 \qquad i \neq j$$

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where $\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j = \delta_{ij}$. The action of the Weyl reflections on an arbitrary weight $\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \Sigma_i \, \omega_i \boldsymbol{\lambda}_i$, is given by

$$S_{\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j} \boldsymbol{\omega} = \boldsymbol{\omega} - (\omega_i - \omega_j)(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j) \tag{1}$$

$$S_{\boldsymbol{\lambda}_i + \boldsymbol{\lambda}_j} \boldsymbol{\omega} = \boldsymbol{\omega} - (\omega_i + \omega_j)(\boldsymbol{\lambda}_i + \boldsymbol{\lambda}_j). \tag{2}$$

We can write the weights of E_8 irreps in the basis used for D_8. The non-zero weights of the adjoint representation (i.e. the roots) are then

$$\begin{aligned} &\pm \boldsymbol{\lambda}_i \pm \boldsymbol{\lambda}_j \qquad i, j = 1, 2, \ldots, 8 \qquad i \neq j \\ &\tfrac{1}{2} \sum_{j=1}^{8} \sigma_j \boldsymbol{\lambda}_j \end{aligned} \tag{3}$$

where $\sigma_j = \pm 1$ and the number of negative σ_j in the sum is odd. The Weyl reflections are given by (1) and (2) together with

$$S_{\frac{1}{2} \Sigma_j \sigma_j \boldsymbol{\lambda}_j} \boldsymbol{\omega} = \boldsymbol{\omega} - \left(\frac{1}{2} \sum_{j=1}^{8} \sigma_j \omega_j \right) \left(\frac{1}{2} \sum_{k=1}^{8} \sigma_k \omega_k \right). \tag{4}$$

Note that two equivalent weights have the same length.

We label the irreps by a set of eight integers $(a_1, \ldots, a_8)$ as given in McKay and Patera (1981). To simplify the notation, we shall write only the non-zero a's with a subscript to indicate their position. For example, $(20010000) = (2_1 1_4)$. Also, to avoid confusion, the E_8 irreps will be enclosed in square brackets, [ ], and the D_8 irreps in round ones, ( ).

The length of a representation $L[\phi]$ is defined as the square of the length of the highest weight. In the orthonormal basis it is simple to find the length of any representation. If $\boldsymbol{\alpha}_j$ are simple roots of E_8 and are expressed in the $\boldsymbol{\lambda}_i$ basis as in figure 1, then the highest weights $\boldsymbol{\pi}_i$ of the basic irreps $[1_i]$ satisfy the relation $(2\boldsymbol{\pi}_i \cdot \boldsymbol{\alpha}_j)/(\boldsymbol{\alpha}_j^2) = \delta_{ij}$. For example, we find $\boldsymbol{\pi}_2 = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 - 2\boldsymbol{\lambda}_8$ and therefore $L[1_2] = 6$. The lengths and dimensions of E_8 and D_8 irreps required in this study are given in tables 1 and 2 (see also Freudenthal 1954, 1956). If we apply the transformation $S_{\boldsymbol{\alpha}}$ on the respective maximal weights of the E_8 irreps of length N, we obtain all the D_8 weights of length N. We will therefore use the following rule.

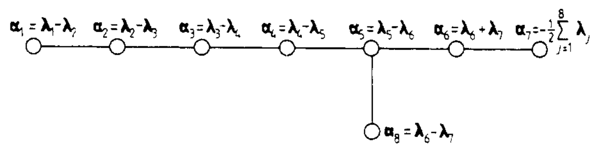

Figure 1. Dynkin diagram for E_8. The simple roots $\boldsymbol{\alpha}_i$ are expressed in the $\boldsymbol{\lambda}_i$ basis.

The direct sum of all E_8 irreps of length N branches to all† D_8 irreps of length N plus some irreps of smaller length. Furthermore, each of these irreps of length N occurs only once in this sum. We may justify this by noting that all the weights of length N which are highest weights of D_8 irreps are equivalent to the highest weight of some E_8 irreps. Therefore these weights will have multiplicity one.

† In fact only 'even' representations of D_8 will appear, where we define an even (odd) irrep of D_8 according to whether $n_1 + 2n_2 \ldots + 8n_8$ is even (odd) where the D_8 representation is $(n_1, n_2, \ldots, n_8)$. This follows from an examination of the basic weights of E_8 in the $\boldsymbol{\lambda}_i$ basis. The sum of the coefficients of each of these weights is always even.

Table 1. E_8 irreducible representations.

Label	$L[\phi]$	Dimension
$[1_1]$	2	248
$[1_7]$	4	3 875
$[1_2]$	6	30 380
$[2_1]$	8	27 000
$[1_8]$	8	147 250
$[1_1 1_7]$	10	779 247
$[1_3]$	12	2 450 240
$[1_1 1_2]$	14	4 096 000
$[1_6]$	14	6 696 000
$[2_7]$	16	4 881 384
$[1_1 1_8]$	16	26 411 008
$[3_1]$	18	1 763 125
$[1_2 1_7]$	18	76 271 625
$[2_1 1_7]$	20	70 680 000
$[1_4]$	20	146 325 270

Table 2. D_8 irreducible representations.

Label	$L[\phi]$	Dimension	Label	$L[\phi]$	Dimension
(0)	0	1	$(1_2 2_7)$	14	595 595
(1_2)	2	120	$(2_1 1_6)$	14	850 850
(1_8)	2	128	$(1_3 1_5)$	14	1 336 608
(2_1)	4	135	$(1_1 1_4 1_7)$	14	2 036 736
(1_4)	4	1 820	(4_1)	16	3 740
$(1_1 1_7)$	4	1 920	$(2_7 1_8)$	16	439 296
$(1_1 1_3)$	6	7 020	$(2_1 2_7)$	16	700 128
(1_6)	6	8 008	(2_4)	16	771 120
$(1_2 1_8)$	6	13 312	$(2_1 1_2 1_8)$	16	898 560
(2_2)	8	5 304	$(1_2 1_3 1_7)$	16	3 294 720
(2_7)	8	6 435	$(1_1 1_2 1_5)$	16	3 686 400
$(2_1 1_8)$	8	15 360	$(1_3 1_7 1_8)$	16	4 084 080
$(1_3 1_7)$	8	56 320	$(1_1 1_5 1_8)$	16	4 264 960
$(1_1 1_5)$	8	60 060	(3_2)	18	129 675
$(2_1 1_2)$	10	8 925	(3_8)	18	183 040
$(1_2 1_4)$	10	141 372	$(3_1 1_3)$	18	255 255
$(1_1 1_2 1_7)$	10	141 440	$(2_1 1_3 1_7)$	18	4 523 904
$(1_4 1_8)$	10	161 280	$(1_1 1_3 1_4)$	18	4 972 500
$(1_1 1_7 1_8)$	10	162 162	$(1_1 1_6 1_7)$	18	6 223 360
(2_3)	12	89 760	$(1_4 1_6)$	18	6 683 040
$(2_1 1_4)$	12	176 800	$(1_1 1_2 1_7 1_8)$	18	10 649 600
$(1_5 1_7)$	12	326 144	$(1_2 1_4 1_8)$	18	11 197 440
$(1_1 1_3 1_8)$	12	670 208	$(2_1 2_2)$	20	260 832
$(1_2 1_6)$	12	716 040	$(1_1 2_2 1_7)$	20	4 426 240
$(3_1 1_7)$	14	87 040	$(2_2 1_4)$	20	4 514 400
$(1_1 1_2 1_3)$	14	344 064	$(1_1 1_7 2_8)$	20	5 940 480
$(1_6 1_8)$	14	465 920	$(1_4 2_8)$	20	6 077 500
$(2_2 1_8)$	14	524 160			

In general, if we can resolve the Kronecker products for both the algebra H and its subalgebra h, we can determine the branching rules $H \to h$. In the case of E_8, however, when we take the Kronecker products of representations for which the branching rules are known, it is usually true that we can deduce directly only the branching rules for pairs of additional E_8 irreps. More information can be obtained by comparing the results from two different Kronecker products. To determine the branching rules completely, it is necessary to equate the dimensions of the E_8 irrep with the sum of the dimensions of D_8 irreps. The method is best illustrated by a few examples. From table 2, we see that if we require that the D_8 irreps into which $[1_1]$ ($[1_7]$) branches be of length $\leqslant 2$ ($\leqslant 4$) we will get

$$[1_1] \to (1_2) + (1_8) \tag{5}$$

$$[1_7] \to (1_1 1_7) + (1_4) + (2_1). \tag{6}$$

In table 3 the Kronecker products for some E_8 irreps are listed. These results have been taken from Wybourne (1979). Table 3(a) gives the results of multiplying $[1_1]$ by the irreps which label the columns. The entries in the table give the multiplicity of each irrep (labelled by the rows) into which the product decomposes. Table 3(b) gives the same information for $[1_7]$. A Kronecker square like $[1_1] \times [1_1]$ can be separated into a symmetric and an antisymmetric part and this is indicated in the tables by an s or a subscript. For example, the Kronecker square of the lowest E_8 irrep is

$$[1_1] \times [1_1] = \{[2_1] + [1_7] + [0]\}_s + ([1_2] + [1_1])_a. \tag{7}$$

Using equation (5), we can also write

$$\begin{aligned} [1_1] \times [1_1] &= \{(1_2) + (1_8)\} \times \{(1_2) + (1_8)\} \\ &= \{(2_8) + 2 \times (1_4) + 2 \times (0) + (2_2) + (2_1)\}_s \\ &\quad + ((1_6) + 2 \times (1_2) + (1_1 1_3))_a + 2 \times (1_1 1_8) + 2 \times (1_1 1_7) + 2 \times (1_8) \end{aligned} \tag{8}$$

Table 3. E_8 Kronecker products.

	(*a*)					(*b*)		
	$[1_1]\times$ $[1_1]$	$[1_7]$	$[1_2]$	$[2_1]$	$[1_8]$	$[1_7]\times$ $[1_7]$	$[1_2]$	$[2_1]$
$[0]$	1_s	0	0	0	0	1_s	0	0
$[1_1]$	1_a	1	1	1	0	1_a	1	0
$[1_7]$	1_s	1	1	0	1	1_s	1	1
$[1_2]$	1_a	1	1	1	1	1_a	2	1
$[2_1]$	1_s	0	1	1	0	1_s	1	1
$[1_8]$		1	1	0	1	1_s	1	1
$[1_1 1_7]$		1	1	1	1	1_a	2	1
$[1_3]$			1	0	1	1_s	1	1
$[1_1 1_2]$			1	1	0	0	1	1
$[1_6]$				0	1	1_a	1	0
$[2_7]$				0	0	1_s	0	0
$[1_1 1_8]$				0	1		1	1
$[3_1]$				1			0	0
$[1_2 1_7]$							1	0
$[2_1 1_7]$								1

Table 4. D_8 Kronecker products.

$(1_2)\times$	(1_2)	(1_8)	(2_1)	(1_4)	(1_11_7)	(1_11_3)	(1_6)	(1_21_8)	(2_2)	(2_8)	(2_11_8)	(1_31_7)	(1_11_5)	(2_11_2)
(0)	1_s	0	0	0	0	0	0	0	0	0	0	0	0	0
(1_2)	1_a	0	1	1	0	1	0	0	1	0	0	0	0	0
(1_8)	0	1	0	0	1	0	0	1	0	0	0	0	0	0
(2_1)	1_s	0	1	0	0	1	0	0	0	0	0	0	0	1
(1_4)	1_s	0	0	1	0	1	1	0	0	0	0	0	1	0
(1_11_7)	0	1	0	0	2	0	0	1	0	0	1	1	0	0
(1_11_3)	1_a	0	1	1	0	2	0	0	1	0	0	0	1	1
(1_6)	0	0	0	1	0	0	1	0	0	1	0	0	1	0
(1_21_8)	0	1	0	0	1	0	0	2	0	0	1	1	0	0
(2_2)	1_s		0	0	0	1	0	0	1	0	0	0	0	1
(2_7)			0	0	0	0	1	0	0	0	0	0	0	0
(2_8)			0	0	0	0	1	0	0	1	0	0	0	0
(2_11_8)			0	0	1	0	0	1	0	0	2	0	0	0
(1_31_7)			0	0	1	0	0	1	0	0	0	2	0	0
(1_11_5)			0	1	0	1	1	0	0	0	0	0	2	0
(2_11_2)			1	0	0	1	0	0	1	0	0	0	0	2
(1_21_4)				1	0	1	0	0	1	0	0	0	1	0
$(1_11_21_7)$					1	0	0	1	0	0	1	1	0	0
(1_41_8)						0	0	1	0	0	0	1	0	0
$(1_11_71_8)$						0	1	0	0	1	0	0	1	0
(2_3)						1	0	0	0	0	0	0	0	0
(2_11_4)						1	0	0	0	0	0	0	1	1
(1_51_7)						0	0	0	0	0	0	1	0	0
$(1_11_31_8)$						0	0	1	0	0	1	1	0	0
(1_21_6)						0	1	0	0	0	0	0	1	0
(3_11_7)						0		0	0	0	1	0	0	0
$(1_11_21_3)$						1		0	1	0	0	0	0	1
(1_61_8)								0	0	0	0	0	0	0
(2_21_8)								1	0	0	0	0	0	0
(1_22_7)									0	0	0	0	0	0
(1_22_8)									0	1	0	0	0	0
(2_11_6)									0		0	0	1	0
(1_31_5)									0		0	0	1	0
$(1_11_41_7)$									0		0	1	0	0
(4_1)									0		0	0	0	1
(2_71_8)									0		0	0	0	0
(2_12_7)									0		0	0	0	0
(2_12_8)									0		0	0	0	0
(2_4)									0		0	0	0	0
$(2_11_21_8)$									0		1	0	0	0
$(1_21_31_7)$									0			1	0	0
$(1_11_21_5)$									0				1	0
$(1_31_71_8)$									0					0
$(1_11_51_8)$									0					0

Table 4. (continued)

	$(1_2)\times$	(1_2)	(1_8)	(2_1)	(1_4)	$(1_1 1_7)$	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$	(2_2)	(2_8)	$(2_1 1_8)$	$(1_3 1_7)$	$(1_1 1_5)$	$(2_1 1_2)$
(3_2)										1					0
(3_8)															0
$(3_1 1_3)$															1
$(2_1 1_3 1_7)$															0
$(1_1 1_3 1_4)$															0
$(1_1 1_6 1_7)$															0
$(1_4 1_6)$															0
$(1_1 1_2 1_7 1_8)$															0
$(1_2 1_4 1_8)$															0
$(2_1 2_2)$															1
$(1_1 2_2 1_7)$															
$(2_2 1_4)$															
$(1_1 1_7 2_8)$															
$(1_4 2_8)$															

	$(2_1)\times$	(2_1)	(1_4)	$(1_1 1_7)$	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$	(2_2)	(2_8)
(0)		1_s	0	0	0	0	0	0	0
(1_2)		1_a	0	0	1	0	0	0	0
(1_8)		0	0	1	0	0	0	0	0
(2_1)		1_s	0	0	0	0	0	1	0
(1_4)		0	1	0	1	0	0	0	0
$(1_1 1_7)$		0	0	1	0	0	1	0	0
$(1_1 1_3)$		0	1	0	2	0	0	1	0
(1_6)		0	0	0	0	1	0	0	0
$(1_2 1_8)$		0	0	1	0	0	1	0	0
(2_2)		1_s	0	0	1	0	0	1	0
(2_7)		0	0	0	0	0	0	0	0
(2_8)		0	0	0	0	0	0	0	1
$(2_1 1_8)$		0	0	1	0	0	1	0	0
$(1_3 1_7)$		0	0	0	0	0	1	0	0
$(1_1 1_5)$		0	1	0	0	1	0	0	0
$(2_1 1_2)$		1_a	0	0	1	0	0	1	0
$(1_2 1_4)$		0	0	0	1	0	0	0	0
$(1_1 1_2 1_7)$		0	0	1	0	0	1	0	0
$(1_4 1_8)$		0	0	0	0	0	0	0	0
$(1_1 1_7 1_8)$		0	0	0	0	1	0	0	1
(2_3)		0	0	0	0	0	0	1	0
$(2_1 1_4)$		0	1	0	1	0	0	0	0
$(1_5 1_7)$		0		0	0	0	0	0	0
$(1_1 1_3 1_8)$		0		0	0	0	1	0	0
$(1_2 1_6)$		0		0	0	0	0	0	0

Table 4. (continued)

	$(2_1)\times$	(2_1)	(1_4)	$(1_1 1_7)$	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$	(2_2)	(2_8)
$(3_1 1_7)$		0		1	0	0	0	0	0
$(1_1 1_2 1_3)$		0			1	0	0	1	0
$(1_6 1_8)$		0			0	0	0	0	0
$(2_2 1_8)$		0			0	0	0	0	0
$(1_2 2_7)$		0			0	0	0	0	0
$(1_2 2_8)$		0			0	0	0	0	0
$(2_1 1_6)$		0			0	1	0	0	0
$(1_3 1_5)$		0			0		0	0	0
$(1_1 1_4 1_7)$		0			0		0	0	0
(4_1)		1_s			0		0	0	0
$(2_7 1_8)$					0		0	0	0
$(2_1 2_7)$					0		0	0	0
$(2_1 2_8)$					0		0	0	1
(2_4)					0		0	0	
$(2_1 1_2 1_8)$					0		1	0	
$(1_2 1_3 1_7)$					0			0	
$(1_1 1_2 1_5)$					0			0	
$(1_3 1_7 1_8)$					0			0	
$(1_1 1_5 1_8)$					0			0	
(3_2)					0			0	
(3_8)					0			0	
$(3_1 1_3)$					1			0	
$(2_1 1_3 1_7)$								0	
$(1_1 1_3 1_4)$								0	
$(1_1 1_6 1_7)$								0	
$(1_4 1_6)$								0	
$(1_1 1_2 1_7 1_8)$								0	
$(1_2 1_4 1_8)$								0	
$(2_1 2_2)$								1	
$(1_1 2_2 1_7)$									
$(2_2 1_4)$									
$(1_1 1_7 2_8)$									
$(1_4 2_8)$									

	$(1_8)\times$	(1_8)	(2_1)	(1_4)	$(1_1 1_7)$	(2_2)	(2_7)	(2_8)	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$	$(2_1 1_8)$	$(1_3 1_7)$	$(1_1 1_5)$
(0)		1_a	0	0	0	0	0	0	0	0	0	0	0	0
(1_2)		1_a	0	0	1	0	0	0	0	0	1	0	0	0
(1_8)		0	0	1	0	0	0	1	0	1	0	0	0	0
(2_1)		0	0	0	1	0	0	0	0	0	0	1	0	0
(1_4)		1_s	0	0	1	0	0	0	0	0	1	0	1	0
$(1_1 1_7)$		0	1	1	0	0	1	0	1	1	0	0	0	1

Table 4. (continued)

$(1_8)\times$	(1_8)	(2_1)	(1_4)	$(1_1 1_7)$	(2_2)	(2_7)	(2_8)	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$	$(2_1 1_8)$	$(1_3 1_7)$	$(1_1 1_5)$
$(1_1 1_3)$	0	0	0	1	0	0	0	0	0	1	1	1	0
(1_6)	1_a	0	0	1	0	0	0	0	0	1	0	1	0
$(1_2 1_8)$	0	0	1	0	1	0	1	1	1	0	0	0	1
(2_2)	0	0	0	0	0	0	0	0	0	1	0	0	0
(2_7)	0	0	0	1	0	0	0	0	0	0	0	1	0
(2_8)	1_s	0	0	0	0	0	0	0	0	1	0	0	0
$(2_1 1_8)$		1	0	0	0	0	0	1	0	0	0	0	1
$(1_3 1_7)$			1	0	0	1	0	1	1	0	0	0	1
$(1_1 1_5)$			0	1	0	0	0	0	0	1	1	1	0
$(2_1 1_2)$			0	0	0	0	0	0	0	0	1	0	0
$(1_2 1_4)$			0	0	0	0	0	0	0	1	0	1	0
$(1_1 1_2 1_7)$			0	0	1	0	0	1	0	0	0	0	1
$(1_4 1_8)$			1	0	0	0	1	0	1	0	0	0	1
$(1_1 1_7 1_8)$				1	0	0	0	0	0	1	1	1	0
(2_3)					0	0	0	0	0	0	0	1	0
$(2_1 1_4)$					0	0	0	0	0	0	1	0	0
$(1_5 1_7)$					0	1	0	0	1	0	0	0	1
$(1_1 1_3 1_8)$					0	0	0	1	0	0	0	0	1
$(1_2 1_6)$					0	0	0		0	1	0	1	0
$(3_1 1_7)$					0	0	0		0	0	0	0	0
$(1_1 1_2 1_3)$					0	0	0		0	0	0	0	0
$(1_6 1_8)$					0	0	1		1	0	0	0	0
$(2_2 1_8)$					1	0	0			0	0	0	0
$(1_2 2_7)$						0	0			0	0	1	0
$(1_2 2_8)$						0	0			1	0	0	0
$(2_1 1_6)$						0	0				1	0	0
$(1_3 1_5)$						0	0				0	1	0
$(1_1 1_4 1_7)$						0	0				0	0	1
(4_1)						0	0				0	0	0
$(2_7 1_8)$						1	0				0	0	0
$(2_1 2_7)$							0				0	0	0
$(2_1 2_8)$							0				1	0	0
(2_4)							0					0	0
$(2_1 1_2 1_8)$							0					0	0
$(1_2 1_3 1_7)$							0					0	0
$(1_1 1_2 1_5)$							0					0	0
$(1_3 1_7 1_8)$							0					1	0
$(1_1 1_5 1_8)$							0						1
(3_2)							0						
(3_8)							1						
$(3_1 1_3)$													
$(2_1 1_3 1_7)$													
$(1_1 1_3 1_4)$													
$(1_1 1_6 1_7)$													
$(1_4 1_6)$													
$(1_1 1_2 1_7 1_8)$													
$(1_2 1_4 1_8)$													

Table 4. (continued)

$(1_8)\times$	(1_8)	(2_1)	(1_4)	(1_11_7)	(2_2)	(2_7)	(2_8)	(1_11_3)	(1_6)	(1_21_8)	(2_11_8)	(1_31_7)	(1_11_5)
(2_12_2)													
$(1_12_21_7)$													
(2_21_4)													
$(1_11_72_8)$													
(1_42_8)													

	$(1_4)\times$ (1_4)	(1_11_7)	(2_2)	(2_8)	(1_11_3)	(1_6)	(1_21_8)	$(1_11_7)\times$ (1_11_7)	(2_2)	(2_8)	(1_11_3)	(1_6)	(1_21_8)
(0)	1_s	0	0	0	0	0	0	1_s	0	0	0	0	0
(1_2)	1_a	0	0	0	1	1	0	2_a	0	0	0	0	1
(1_8)	0	1	0	0	0	0	1	0	0	0	1	1	0
(2_1)	1_s	0	0	0	1	0	0	1_s	0	0	0	0	1
(1_4)	1_s	0	1	1	1	1	0	2_s	0	0	0	0	2
(1_11_7)	0	2	0	0	0	0	2	0	1	1	2	2	0
(1_11_3)	1_a	0	1	0	2	1	0	2_a	0	0	0	0	3
(1_6)	1_a	0	0	1	1	2	0	2_a	0	0	0	0	2
(1_21_8)	0	2	0	0	0	0	3	0	1	1	3	2	0
(2_2)	1_s	0	1	0	1	0	0	1_s	0	0	0	0	1
(2_7)	1_s	0	0	0	0	1	0	1_s	0	0	0	0	1
(2_8)	1_s	0	0	1	0	1	0	1_s	0	0	0	0	1
(2_11_8)	0	1	0	0	0	0	1	0	1	1	2	1	0
(1_31_7)	0	2	0	0	0	0	2	0	1	1	2	2	0
(1_11_5)	1_s	0	1	1	2	1	0	$2_{a,s}$	0	0	0	0	3
(2_11_2)	0	0	0	0	1	0	0	1_a	0	0	0	0	1
(1_21_4)	1_a	0	1	0	2	1	0	1_a	0	0	0	0	2
$(1_11_21_7)$	0	1	0	0	0	0	2	0	2	0	3	1	0
(1_41_8)	0	1	0	0	0	0	2	0	0	1	1	2	0
$(1_11_71_8)$	1_a	0	0	1	1	2	0	$2_{a,s}$	0	0	0	0	3
(2_3)	1_s	0	0	0	1	0	0	0	0	0	0	0	1
(2_11_4)	0	0	1	0	1	0	0	1_s	0	0	0	0	1
(1_51_7)	0	1	0	0	0	0	1	0	0	1	0	2	0
$(1_11_31_8)$	0	1	0	0	0	0	2	0	1	1	2	1	0
(1_21_6)	1_s	0	1	1	1	1	0	1_s	0	0	0	0	2
(3_11_7)	0	0	0	0	0	0	0	0	0	0	1	0	0
$(1_11_21_3)$	0	0	1	0	1	0	0	0	0	0	0	0	1
(1_61_8)	0	0	0	0	0	0	1	0	0	1	0	1	0
(2_21_8)	0	0	0	0	0	0	1	0	1	0	1	0	0
(1_22_7)	0	0	0	0	0	1	0	1_a	0	0	0	0	1
(1_22_8)	0	0	0	1	0	1	0	0	0	0	0	0	1
(2_11_6)	0	0	0	0	1	0	0	1_a	0	0	0	0	1
(1_31_5)	1_a	0	0	0	1	1	0	0	0	0	0	0	1
$(1_11_41_7)$	0	1	0	0	0	0	1	0	0	0	1	1	0

Table 4. (continued)

	$(1_4)\times$ (1_4)	$(1_1 1_7)$	(2_2)	(2_8)	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$	$(1_1 1_7)\times$ $(1_1 1_7)$	(2_2)	(2_8)	$(1_1 1_3)$	(1_6)	$(1_2 1_8)$
(4_1)	0		0	0	0	0	0	0	0	0	0	0	0
$(2_7 1_8)$	0		0	0	0	0	0	0	0	1	0	1	0
$(2_1 2_7)$	0		0	0	0	0	0	1_s	0	0	0	0	0
$(2_1 2_8)$	0		0	0	0	0	0		0	0	0	0	1
(2_4)	1_s		0	0	0	0	0		0	0	0	0	0
$(2_1 1_2 1_8)$			0	0	0	0	0		1	0	1	0	0
$(1_2 1_3 1_7)$			0	0	0	0	1		1	0	1	0	0
$(1_1 1_2 1_5)$			1	0	1	0	0		0	0	0	0	1
$(1_3 1_7 1_8)$			0	1	0	1	0		0	0	0	0	1
$(1_1 1_5 1_8)$			0	0	0	0	1		0	1	0	1	0
(3_2)			0	0	0	0	0		0	0	0	0	0
(3_8)			0	0	0	0	0		0	0	0	0	0
$(3_1 1_3)$			0	0	0	0	0		0	0	0	0	0
$(2_1 1_3 1_7)$			0	0	0	0	0		0	0	1	0	0
$(1_1 1_3 1_4)$			0	0	1	0	0		0	0		0	0
$(1_1 1_6 1_7)$			0	0		0	0		0	0		1	0
$(1_4 1_6)$			0	0		1	0		0	0			0
$(1_1 1_2 1_7 1_8)$			0	0			0		0	0			1
$(1_2 1_4 1_8)$			0	0			1		0	0			
$(2_1 2_2)$			0	0					0	0			
$(1_1 2_2 1_7)$			0	0					1	0			
$(2_2 1_4)$			1	0						0			
$(1_1 1_7 2_8)$				1						1			
$(1_4 2_8)$				1									

where the D_8 Kronecker products are given in table 4. These were calculated using Young's tableau (see Fischler 1981). Table 4 is arranged in the same way as table 3 and each section of the table corresponds to a product by the irrep indicated in the upper left-hand corner. After subtracting the irreps belonging to $[1_7]$, $[0]$ and $[1_1]$, we find

$$[2_1]+[1_2]\rightarrow(2_8)_s+(2_2)_s+2\times(1_2 1_8)+(1_1 1_3)_a+(1_6)_a+(1_1 1_7)+(1_4)_s+(1_8)+(1_2)_a+(0)_s. \tag{9}$$

From the symmetry property and using the rule that the sum of E_8 irreps of length N branches into all D_8 irreps of the same length, we get

$$[2_1]\rightarrow(2_8)+(1_2 1_8)+(2_2)+(1_4)+(0)+\ldots \tag{10}$$

$$[1_2]\rightarrow(1_2 1_8)+(1_1 1_3)+(1_6)+(1_2)+\ldots. \tag{11}$$

Using the law of dimensions, the branching rule is immediately completed. The result is found in table 5, where the branching multiplicities of the D_8 irreps labelling the rows are the entries of a given column labelled by an E_8 irrep.

In order to illustrate the added complexities of obtaining branching rules for higher-dimensional irreps, we give a second example. If we assume that we have found the branching rules for all irreps of length less than 14 and also for $[1_1 1_2]$ of length 14, from table 3(*b*) we see that the Kronecker product $[1_7]\times[1_7]$ contains only two irreps for which the branching rules are unknown. These are $[2_7]$ and $[1_6]$. In the Kronecker product $[1_8]\times[1_1]$, the unknown branching rules are for $[1_11_8]$ and $[1_6]$. After taking the respective products in D_8, using table 4, and subtracting the irreps for which the branching rules have already been calculated (table 5), we get

$$\begin{aligned}[2_7]+[1_6]\to{}&(2_12_7)+(2_4)+(4_1)+2\times(3_11_7)\\&+2\times(1_11_41_7)+(1_22_7)+(1_31_5)+(2_11_6)+(1_11_31_8)+(1_51_7)\\&+(1_21_6)+2\times(2_11_4)+(1_11_71_8)+2\times(1_11_21_7)+2\times(1_41_8)\\&+(2_11_2)+(1_21_4)+(2_11_8)+(1_31_7)+(2_8)+(2_2)+2\times(1_11_5)\\&+2\times(1_21_8)+(1_11_3)+(1_6)+(1_11_7)+(1_4)+(1_8)+(1_2)+(0)\end{aligned}\qquad(12)$$

$$\begin{aligned}[1_11_8]+[1_6]\to{}&(1_21_31_7)+(2_12_8)+(2_71_8)+(2_11_21_8)+(1_31_71_8)+(1_11_51_8)+(1_11_21_5)\\&+(3_11_7)+2\times(1_11_41_7)+2\times(1_22_7)+(1_11_21_3)\\&+2\times(1_31_5)+2\times(2_11_6)+3\times(1_11_31_8)+2\times(1_51_7)\\&+(1_21_6)+2\times(2_11_4)+(2_3)+3\times(1_11_71_8)+3\times(1_11_21_7)+(1_41_8)\\&+2\times(2_11_2)+2\times(1_21_4)+3\times(2_11_8)+3\times(1_31_7)+(2_7)+3\times(1_11_5)\\&+2\times(1_21_8)+3\times(1_11_3)+2\times(1_6)+3\times(1_11_7)+(2_1)+(1_4)+(1_2).\end{aligned}\qquad(13)$$

All the irreps of length 16 in equation (12) or (13) must belong to $[2_7]$ or $[1_11_8]$ respectively. All those of length 14 which are not in $[1_11_2]$ appear only once in $[1_6]$ and the rest are in $[2_7]$ and $[1_11_8]$. Furthermore, all symmetric irreps are in $[2_7]$ and the antisymmetric ones in $[1_6]$. Also, every irrep which appears in (12) but not in (13) must belong to $[2_7]$. Also each irrep which is in (13) but not in (12) must be in $[1_11_8]$. Therefore

$$\begin{aligned}[2_7]\to{}&(4_1)+(2_4)+(2_12_7)+(3_11_7)+(1_11_41_7)+(1_21_6)\\&+(2_8)+(2_2)+(0)+(1_41_8)+(1_8)+\dots\end{aligned}\qquad(14)$$

$$\begin{aligned}[1_6]\to{}&(3_11_7)+(1_11_41_7)+(1_22_7)+(1_31_5)\\&+(2_11_6)+(1_11_71_8)+(2_11_2)+(1_21_4)+(1_11_3)+(1_2)+\dots\end{aligned}\qquad(15)$$

$$\begin{aligned}[1_11_8]\to{}&(1_21_31_7)+(2_12_8)+(2_71_8)+(2_11_21_8)+(1_31_71_8)+(1_11_51_8)+(1_11_21_5)+(1_11_41_7)\\&+(1_22_7)+(1_11_21_3)+(1_31_5)+(2_11_6)+2\times(1_11_31_8)+(1_51_7)+(1_21_6)\\&+(2_3)+2\times(1_11_71_8)+(1_11_21_7)+(2_11_2)+(1_21_4)\\&+2\times(2_11_8)+2\times(1_31_7)+(2_7)+(1_11_5)+2\times(1_11_3)\\&+(1_6)+2\times(1_11_7)+(2_1)+\dots.\end{aligned}\qquad(16)$$

The sums of the dimensions of the irreps missing in (14), (15) and (16) are respectively 554712 for $[2_7]$ and $[1_11_8]$ and 1469572 for $[1_6]$. By matching those dimensions with the dimensions of the irreps remaining in (12) and (13), we can find a unique solution for the branching rule. The additional branching rules listed in table 5 are derived similarly. This table which gives the branching rules of 14 irrep of E_8 increases by 5 the number given by King and Al-Qubanchi (1981). Using the techniques outlined above, one can enlarge the table. It is the tabulation of the D_8 Kronecker products that presents the greatest difficulty.

Table 5. Branching rules for $E_8 \to D_8$.

L	$[\phi]$	2 $[1_1]$	4 $[1_7]$	6 $[1_2]$	8 $[2_1]$	8 $[1_8]$	10 $[1_1 1_7]$	12 $[1_3]$	14 $[1_1 1_2]$	14 $[1_6]$	16 $[2_7]$	16 $[1_1 1_8]$	18 $[3_1]$	18 $[1_2 1_7]$	20 $[2_1 1_7]$
0	(0)	0	0	0	1	0	0	0	0	0	1	0	0	0	0
2	(1_2)	1	0	1	0	0	1	0	1	1	0	0	1	1	0
	(1_8)	1	0	0	1	0	1	0	1	0	1	0	1	1	0
4	(2_1)		1	0	0	1	0	1	0	0	0	1	0	0	1
	(1_4)		1	0	1	0	1	1	1	0	1	1	0	1	2
	$(1_1 1_7)$		1	1	0	1	1	1	1	1	0	2	0	1	2
6	$(1_1 1_3)$			1	0	1	1	1	1	1	0	2	0	2	1
	(1_6)			1	0	0	1	0	1	1	0	1	1	2	1
	$(1_2 1_8)$			1	1	0	1	1	2	1	1	1	1	3	2
8	(2_2)				1	0	0	1	1	0	1	0	0	1	1
	(2_7)				0	1	0	1	0	0	0	1	0	0	0
	(2_8)				1	0	0	0	1	0	1	0	1	1	2
	$(2_1 1_8)$				0	1	1	1	0	1	0	2	0	1	1
	$(1_3 1_7)$				0	1	1	1	1	1	0	2	0	2	2
	$(1_1 1_5)$				0	1	1	1	1	1	1	2	0	2	2
10	$(2_1 1_2)$						1	0	0	1	0	1	0	1	0
	$(1_2 1_4)$						1	0	1	1	0	1	1	3	1
	$(1_1 1_2 1_7)$						1	1	1	1	1	2	0	3	1
	$(1_4 1_8)$						1	0	1	0	1	1	1	2	2
	$(1_1 1_7 1_8)$						1	1	1	1	0	2	0	2	2
12	(2_3)							1	0	0	0	1	0	0	1
	$(2_1 1_4)$							1	0	1	1	1	0	1	1
	$(1_5 1_7)$							1	0	1	0	1	0	1	1
	$(1_1 1_3 1_8)$							1	1	1	0	2	0	2	2
	$(1_2 1_6)$							1	1	0	1	1	0	2	2
14	$(3_1 1_7)$								0	1	1	0	0	1	0
	$(1_1 1_2 1_3)$								1	0	0	1	0	1	1
	$(1_6 1_8)$								1	0	0	0	0	1	1
	$(2_2 1_8)$								1	0	0	0	1	1	1
	$(1_2 2_7)$								0	1	0	1	0	1	0
	$(1_2 2_8)$								1	0	0	0	1	1	1
	$(2_1 1_6)$								0	1	0	1	0	2	0
	$(1_3 1_5)$								0	1	0	1	0	1	1
	$(1_1 1_4 1_7)$								0	1	1	1	0	2	1
16	(4_1)										1	0	0	0	0
	$(2_7 1_8)$										0	1	0	0	0
	$(2_1 2_7)$										1	0	0	1	0
	$(2_1 2_8)$										0	1	0	0	1
	(2_4)										1	0	0	0	1
	$(2_1 1_2 1_8)$										0	1	0	1	1
	$(1_2 1_3 1_7)$										0	1	0	1	1
	$(1_1 1_2 1_5)$										0	1	0	1	1
	$(1_3 1_7 1_8)$										0	1	0	1	1
	$(1_1 1_5 1_8)$										0	1	0	1	1

Table 5. (continued)

$L[\phi]$	2 $[1_1]$	4 $[1_7]$	6 $[1_2]$	8 $[2_1]$	8 $[1_8]$	10 $[1_1 1_7]$	12 $[1_3]$	14 $[1_1 1_2]$	14 $[1_6]$	16 $[2_7]$	16 $[1_4 1_8]$	18 $[3_1]$	18 $[1_2 1_7]$	20 $[2_1 1_7]$
18 (3_2)												1	0	0
(3_8)												1	0	0
$(3_1 1_3)$												0	1	0
$(2_1 1_3 1_7)$												0	1	0
$(1_1 1_3 1_4)$												0	1	0
$(1_1 1_6 1_7)$												0	1	0
$(1_4 1_6)$												0	1	0
$(1_1 1_2 1_7 1_8)$												0	1	1
$(1_2 1_4 1_8)$												0	1	1
20 $(2_1 2_2)$														1
$(1_1 2_2 1_7)$														1
$(2_2 1_4)$														1
$(1_1 1_7 2_8)$														1
$(1_4 2_8)$														1

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